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A New Proof of Torelli's Theorem

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A NEW PROOF OF TORELLI'S THEOREM

Henrik H. Martens[1]

ABSTRACT

Torelli [9] proved that if two closed Riemann surfaces have the same period matrix (with respect to canonical homology bases), then they are conformally equivalent. Thus a period matrix of a closed Riemann surface determines its conformal structure completely, and it is this fact that makes the entries of a period matrix suitables candidates for a set of explicit moduli for closed Riemann surfaces.

Modern proofs of Torelli's theorem, valid in the abstract case, have been given by Weil [10], Andreotti [1], and Matsusaka [5]. All of these proofs make extensive use of the methods of algebraic geometry. The primary object of this paper is to give an analytically formulated proof based on results contained in the standard texts on the theory of Riemann surfaces, and on a theorem of Riemann regarding the vanishing of the first-order theta-function. The analytical formulation of the paper should not, however, be interpreted to mean that our method is restricted to the classical case. The algebraic geometer will find that Sections 2 and 3 contain a complete proof of Torelli's theorem for the abstract case as well.

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[1] CDTP Fellow from the Bell Telephone Laboratories, Incorporated.

The contents of this paper may be summarized as follows. Section 1 contains definitions of the basic concepts involved, as well as a formulation of the result to be proved and a brief survey of some of the classical results in this area of research. Section 2 is devoted to the proofs of certain auxiliary lemmas about subsets of the Jacobian variety of a Riemann surface. Section 3 contains a proof of Torelli's theorem, granting the validity of Riemann's theorem, a proof of which is given in the appendix.

1. INTRODUCTION

Let X be a closed Riemann surface of genus g. Let $\gamma = (\gamma_1, \ldots, \gamma_{2g})$ be any basis for the first homology group on X, and let α be an Abelian differential of the first kind on X. By the period vector of α with respect to γ we shall mean the vector $(\gamma, \alpha) = (a_1, \ldots, a_{2g})$, where

$$a_i = \int_{\gamma_i} \alpha = (\gamma_i, \alpha)$$

Let $\omega = (\omega^1, \ldots, \omega^g)$ be a basis for the Abelian differentials of the first kind on X. By the period matrix of ω with respect to γ we shall mean the $g \times 2g$ matrix $(\gamma, \omega) = (\gamma_i, \omega^j)$ whose j^{th} row vector is the period vector (γ, ω^j).

For the purpose of investigating the properties of period vectors and matrices, it is convenient (and usual) to assume the situation to be normalized by taking γ to be a canonical homology basis. By this is meant that the intersection matrix, $(\gamma_i \circ \gamma_j)$, has the form

$$\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} = J$$

where 0 is the $g \times g$ null-matrix, and E is the $g \times g$ unit matrix. Now, let γ be a canonical homology basis, and let $\bar{\gamma}$ be a homology basis related to γ by

$$\bar{\gamma}_i = \sum m_i^j \gamma_j \ ,$$

where the matrix, $M = \left(m_i^j\right)$, is unimodular with integral entries. An easy calculation shows that $\bar{\gamma}$ is a canonical homology basis if and only if ${}^tMJM = J$. Furthermore, under a change of basis of this kind, we clearly have

$$(\bar{\gamma}, \alpha) = (\gamma, \alpha)M \ ,$$

and

$$(\bar{\gamma}, \omega) = (\gamma, \omega)M \ ,$$

where α is an Abelian differential of the first kind, and ω is a basis for such differentials.

It is shown in standard texts, [8], that if γ is a canonical homology basis, and if ω is a basis for the Abelian differentials of the first kind, then the period matrix of ω with respect to γ has the form

$$(\gamma, \omega) = (B; C) \ ,$$

where B and C are nonsingular $g \times g$ matrices. Hence, by a suitable change of ω , we may assume

$$(\gamma, \omega) = (\pi i E; A) \ , \quad A = \pi i B^{-1} C \ ,$$

where E is the $g \times g$ unit matrix. It is then shown that A must be symmetric and have a negative definite real part. A matrix, $(\pi i E; A)$, where A satisfies these conditions will be referred

to as a Riemann matrix.[1]

Two 2g-vectors, a and a', will be said to be symplectically equivalent iff $a' = aM$, where M is a $2g \times 2g$ unimodular matrix with integral entries, and ${}^tMJM = J$. Two Riemann matrices, $(\pi iE; A)$ and $(\pi iE; A')$ will be said to be symplectically equivalent iff

$$(\pi iE;\ A)M = (B;\ C)$$

and

$$A' = \pi i B^{-1} C\ ,$$

where M satisfies the same conditions. It is clear that if a 2g-vector, a, is the period vector of an Abelian differential of the first kind on a closed Riemann surface with respect to a canonical homology basis, then so is any 2g-vector, a', symplectically equivalent to a, and conversely. A similar statement holds for Riemann matrices.

A precise statement of Torelli's theorem in the classical case may now be given.

Torelli's Theorem: Let X and Y be closed Riemann surfaces of genus g, and let $(\pi iE; A)$ and $(\pi iE; A')$ be period matrices with respect to canonical homology bases on X and Y, respec-

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[1] For simplicity we are here restricting the usual terminology, see [2].

tively. Then, if $(\pi i E; A)$ is symplectically equivalent to $(\pi i E; A')$, X is conformally equivalent to Y .

We may observe here that in the case $g = 1$, Torelli's theorem reduces to a well-known result in the theory of elliptic functions, and we shall here tacitly assume $g \geq 2$.

Torelli's theorem is one of a remarkably limited number of known results about period matrices of closed Riemann surfaces. From the definition of a Riemann matrix, it is easily seen that it depends on $\frac{1}{2}(g+1)g$ free parameters. For $g \geq 2$, the number of free parameters determining a closed Riemann surface of genus g , is known to be $3g - 3$. Hence, for $g \geq 4$, additional conditions must be imposed on a Riemann matrix. Schottky [6] gave the single condition necessary for $g = 4$, but did not show sufficiency. Gerstenhaber [3] observed that a result of Wirtinger implies that A cannot have a row vector with only one nonvanishing entry, if $(\pi i E; A)$ is to be a period matrix with respect to a canonical homology basis on some closed Riemann surface. Weil [10] showed that for $g = 2$ it suffices to exclude such matrices.

For genera higher than 4, the conditions on Z are not yet known, although conditions for the hyperelliptic case apparently have been determined [11]. I do not know whether the last-mentioned results are available in a modern form. For genera higher than 2 , sufficient conditions of a reasonable kind are not yet known.

In the appendix we shall give a simple proof of Wirtinger's result by proving a more general theorem which states that if

$(\pi iE; A)$ is a period matrix of a closed Riemann surface, A cannot be symplectically equivalent to a direct sum of smaller matrices.

Finally, it may be mentioned that the situation with respect to period vectors is much better. Haupt [4] proved that a necessary and sufficient condition that an arbitrary $2g$-vector, a, be a period vector of some Abelian differential of the first kind with respect to a canonical homology basis of some closed Riemann surface of genus g, is that a satisfy certain inequalities derived from the bilinear relations of Riemann, and that a is not symplectically equivalent to a $2g$-vector with only two nonvanishing components.

2. THE JACOBIAN VARIETY OF A CLOSED RIEMANN SURFACE

Let X be a closed Riemann surface of genus g. Let γ be a canonical homology basis on X, and let ω be a basis for the Abelian differentials of the first kind on X such that

$$(\gamma, \omega) = (\pi iE; A) .$$

The column vectors, (γ_i, ω) of (γ, ω) generate a discrete group of translations, Ω, in C^g, whose fundamental region may be taken as the nondegenerate parallelepiped spanned by the column vectors. By the Jacobian Variety of X we shall mean the multitorus

$$J(X) = C^g/\Omega .$$

We observe that the fundamental region of Ω, and hence also $J(X)$, is compact.

There exists a natural embedding of X in $J(X)$ given by

$$u^i(Q) = \int_P^Q \omega^i ,$$

where P is a fixed point on X, and Q varies over X. That this mapping is holomorphic, is trivial. That it is 1-1 follows from the fact that, by Abel's theorem, if $u(Q_1) = u(Q_2)$, Q_1 and Q_2 must be linearly equivalent. This means that there is a meromorphic function on X with a simple pole at Q_1 and a simple zero at Q_2, and no other poles or zeros. But this is known to be impossible. Hence $Q_1 = Q_2$.

By a canonical embedding

$$\phi : X \longrightarrow J(X) ,$$

we shall mean any map, ϕ, which differs from the map $Q \longrightarrow u(Q)$ by a constant. In this paper we shall assume ϕ to be the map defined above.

We extend our definition of ϕ to a mapping

$$\phi : X^r \longrightarrow J(X) ,$$

of the r-fold Cartesian product of X with itself into $J(X)$ by

$$\phi(Q_1, \ldots, Q_r) = \sum \phi(Q_i) ,$$

and define W^r to be the image of X^r under ϕ. It is clear that the sets, W^r, are compact.

It should be observed that the canonically imbedded surface, W^1 , has an induced conformal structure which makes it into a Riemann surface, conformally equivalent to X . To see this, we observe first that, by definition, a function on W^1 is holomorphic at a point iff it is a restriction to W^1 of some function holomorphic in a neighborhood of the point in $J(X)$. Since Φ is a holomorphic map, it follows that every meromorphic function, f , on W^1 yields a meromorphic function, $f \circ \Phi$ on X . If, on the other hand, f is a meromorphic function on X , we form the meromorphic function

$$F(Q_1, \ldots, Q_g) = f(Q_1) + \ldots + f(Q_g)$$

on the g-fold symmetric product of X with itself. This is known to yield a meromorphic function on $J(X)$, $F \circ \Phi^{-1}$. Since $f \circ \Phi^{-1}$ is restriction of the latter to W^1 , it follows that it is a meromorphic function on W^1 . Thus, Φ is biholomorphic, and in order to prove Torelli's theorem, it suffices to prove the corresponding statement about the imbedded image, W^1 .

Let S be a subset of $J(X)$. We shall say that S is irreducible if it has the property that the set of zeros of a (multiplicative) holomorphic function on $J(X)$ either contains S , or else intersects S in a lower-dimensional subset. One sees immediately from this definition that if S is irreducible, and if T is a subset of $J(X)$ defined as the common set of zeros of a (possibly infinite) collection of (multiplicative) holomorphic functions on $J(X)$, then the intersection of S and

T is a lower-dimensional subset of S, unless S is a subset of T.

Lemma 1: The sets W^r, their translates and reflections, are irreducible.

Proof: Let F be a (multiplicative) holomorphic function on $J(X)$, and let f be the restriction of F to W^r. Then $f \circ \Phi$ is a (multiplicative) holomorphic function on the cartesian product, X^r. It is well known that the zeros of this induced function form a lower-dimensional subset of X^r, unless the function vanishes identically. On the other hand, it is easy to see that Φ is generally of maximal rank on X^r, and hence W^r is r-dimensional. But then the image of the zeros of the induced function is a lower-dimensional subset of W^r, and the lemma is proved.

We shall now study some of the properties of, and relations between the sets W^r, their translates and reflections. We observe that the map, $\Phi : X^r - J(X)$ may also be interpreted as a map of the set of positive divisors of degree r, $D = (Q_1 + \ldots + Q_r)$, into $J(X)$, setting

$$\Phi(D) = \Phi(Q_1) + \ldots + \Phi(Q_r) .$$

Now, given any positive divisor, D, of degree $g-1$, we can find an Abelian differential of the first kind which vanishes at the points of D. Hence there exists a positive divisor, D^*, of degree $g-1$, such that $D + D^*$ is a canonical divisor. Since all canonical divisors are linearly equivalent, it follows by Abel's

theorem there is a constant, k, depending only on the choice of P, such that

$$\phi(D + D^*) = k .$$

Hence, for every D of degree $g-1$, there is a D^* of degree $g-1$ such that $\phi(D) = -(\phi(D^*) - k)$.

In order to state this result in a more interesting form, we introduce the following notation: Let S be a subset of $J(X)$. Then we define S_a and $-S_a$ by the conditions

$$u \varepsilon S_a \Longleftrightarrow u-a \varepsilon S ,$$

$$u \varepsilon -S_a \Longleftrightarrow -u \varepsilon S_a ,$$

where a is any point of $J(X)$. Our result now takes the form:

Lemma 2:

$$-W_a^{g-1} = W_{-(a+k)}^{g-1} ,$$

where $k = \phi(Z)$ and Z is any canonical divisor on X.

A similar result does not, in general, hold for W^r, $r < g-1$.

We shall now study some of the relations between the sets W_a^r. The results are stated in the following three lemmas which are believed to be new.

Lemma 3: Let $r \leq t \leq g-1$. Then

$$W_a^r \subseteq W_b^t \Longleftrightarrow a \varepsilon W_b^{t-r} \Longleftrightarrow b \varepsilon -W_{-a}^{t-r} .$$

Proof: We must here interpret W_a^0 as the set whose only member is a . The equivalence on the right is disposed of by observing that if $a \varepsilon W_b^{t-r}$, then $a = \phi(D) + b$, $\deg(D) = t-r$, and hence $b = -(\phi(D) - a)$, which means that $b \varepsilon -W_{-a}^{t-r}$.

To prove the left-hand equivalence, assume that $W_a^r \subset W_b^t$. Then for every positive divisor, D , of degree r , there is a positive divisor, $\hat{D}$, of degree t , such that

$$\phi(D) + a = \phi(\hat{D}) + b .$$

In particular, there is a positive divisor, A , of degree $\leq t$, such that

$$a = \phi(A) + b ,$$

since $a \varepsilon W_a^r$. This representation may not be unique, and we assume that A is chosen to be of the smallest degree, $s \leq t$, for which the equation can be satisfied. Then $\dim(-A) = 1$, since otherwise A would be linearly equivalent to a positive divisor of degree s containing P as a point. But then $\phi(A) = \phi(B+P) = \phi(B)$, for some positive divisor, B of degree $s-1$.

Combining the two equations, we have

$$\phi(D) + \phi(A) = \phi(\hat{D}) .$$

If $r+s > t$, we can choose D of degree $(t+1-s) \leq r$ such that D does not contain P , and $\dim(-D-A) = 1$, and get

$$D + A \sim \hat{D} + P ,$$

where $\sim$ denotes linear equivalence. But this is impossible, since $D + A$ does not contain P, and $\dim(-D-A) = 1$. Hence $s + r \leq t$, and $a \in W_b^{t-r}$. The rest of the proof is trivial.

Let S and T be subsets of $J(X)$. We define the following operations

$$S \oplus T = \cup \{S_a : a \in T\} ,$$

and

$$S \ominus T = \cap \{S_{-a} : a \in T\} .$$

If T is the empty set, $S \ominus T$ should be taken as the empty set.

These operations are not, in general, inverse to each other, although the following results are easily verified:

$$(S \oplus T) \ominus T \supset S ,$$

and

$$(S \ominus T) \oplus T \subset S .$$

We also observe that if $S \subset S'$ and $T' \subset T$, then $S \ominus T \subset S' \ominus T'$. This is an immediate consequence of the definitions.

When these operations are applied to the sets, W_a^r, they behave very regularly, as shown in the following lemma:

Lemma 4: *Let* $0 \leq r \leq t \leq g-1$. *Then*

$$W_a^r \oplus W_b^t = W_{a+b}^{r+t} ,$$

and

$$W_a^t \ominus W_b^r = W_{a-b}^{t-r} .$$

Proof: The first statement is a triviality. The second statement may be proved as follows:

$$u \varepsilon W_{a-b}^{t-r} \Leftrightarrow b-a \varepsilon W_{-u}^{t-r} \Leftrightarrow W_{b-a}^r \subset W_{-u}^t ,$$

by Lemma 3. By a translation, $W_{b-a}^r \subset W_{-u}^t \Leftrightarrow W_b^r \subset W_{a-u}^t$. Let v be any point in W_b^r . If $v \varepsilon W_{a-u}^t$, then $u-a \varepsilon W_{-y}^t$, or equivalently $u \varepsilon W_{a-v}^t$. If this holds for all $v \varepsilon W_b^r$, then $u \varepsilon W_a^t \ominus W_b^r$, and conversely, by the definition of $\ominus$. This completes the proof.

To explain the limitations on r and t , we may observe that if $r \geq g$, $W_a^r = J(X)$, and for any subset $S \subset J(X)$, $J(X) \ominus S = J(X)$.

We observe that while sets of the form $W_a^t \ominus \left(-W_b^r\right)$ may not, in general, be among our W-sets, the following special formulas hold by virtue of Lemma 2:

$$W_a^{g-1} \ominus \left(-W_b^r\right) = -W_{-(a+k+b)}^{g-1-r}$$

$$-W_a^{g-1} \ominus \left(W_b^r\right) = W_{-(a+k+b)}^{g-1-r}$$

The validity of the first formula is shown by

$$W_a^{g-1} \ominus \left(-W_b^r\right) = \cap\left\{W_{a+u}^{g-1};\ u \epsilon W_b^r\right\}$$

$$= \cap\left\{-W^{g-1}_{-(a+k)-u} : u \epsilon W_b^r\right\}$$

$$= -\left(W^{g-1}_{-(a+k)} \ominus W_b^r\right)$$

$$= -W^{g-1-r}_{-(a+k+b)} \quad .$$

Finally, let S be a dense subset of W_b^r . Then $W_a^t \ominus S = W_{a-b}^{t-r}$. This can be seen as follows: If $v \epsilon W_a^t \ominus S$, then $v \epsilon W_{a-s}^t$ for every $s \epsilon S$. Let w be a limit point of S, not belonging to S . Since $v \epsilon W_{a-s}^t \langle = \rangle a-s \epsilon -W_{-v}^t$, and since the latter set is compact, it follows that $a-w \epsilon -W_{-v}^t$. But then $v \epsilon W_{a-w}^t$ for every limit point, w , of S . Hence $v \epsilon W_a^t \ominus W_b^r$.

We shall now consider intersections of the form $W_a^{g-1} \cap W_b^{r+1}$, where $r+1 \leq g-1$. For generally chosen a and b , such intersections are difficult to characterize. However, it is an easy consequence of Lemma 3, that

$$W_c^r \subset W_a^{r+1} \cap W_b^{g-1} \quad \text{if} \quad c \epsilon W_a^1 \cap W_b^{g-1-r} \quad .$$

Lemma 5: *Given two positive divisors*, R *and* $\hat{R}$, *of degrees* 1 *and* $(g-1-r)$, *respectively, and constants*, a , b *and* c *such*

that

$$c = \Phi(R) + a = \Phi(\hat{R}) + b \quad ,$$

then, unless $\quad W_a^{r+1} \cap W_b^{g-1} = W_a^{r+1} \quad ,$

we have $\quad W_a^{r+1} \cap W_b^{g-1} = W_c^{r} \cup S \quad ,$

where $\quad S = W_a^{r+1} \cap -W^{g-2}_{-(a+k-\Phi(\hat{R}))} \quad .$

Proof: Suppose R is a point of $\hat{R}$. Then $\Phi(R) + a = \Phi(\hat{R}) + b$ yields an equation $a = \Phi(R') + b$, $\deg(R') = g-2-r$. But then $a \in W_b^{g-2-r}$ and $W_a^{r+1} \subset W_b^{g-1}$. Hence we may assume that R is not a point of $\hat{R}$.

Let u be any point of the intersection. Then there exist positive divisors, D and $\hat{D}$, of degrees $(r+1)$ and $(g-1)$, respectively, such that

$$u = \Phi(D) + a = \Phi(\hat{D}) + b \quad .$$

Substituting for a, we have

$$\Phi(D) + \Phi(\hat{R}) = \Phi(\hat{D}) + \Phi(R) \quad ,$$

or

$$D+\hat{R} \sim \hat{D}+R \quad .$$

If this is an identity, R must occur on the left-hand side, and since R is not a point of $\hat{R}$, it must be a point of D. Hence

$$u = \Phi(D) + a = \Phi(D') + \Phi(R) + a = \Phi(D') + c$$

and since $\deg(D') = r$, $u \in W^r_c$.

If the linear equivalence is not an identity, then $\dim(-D-\hat{R}) \geq 2$, and we can find a divisor B of degree g , containing an arbitrary point, $Q \in X$, such that $D+\hat{R} \sim B = B'+Q$. But then $u = \Phi(D) + a = \Phi(B') + \Phi(Q) - \Phi(\hat{R}) + a$ or $u \in W^{g-1}_{a-\Phi(\hat{R})+\Phi(Q)}$ for any point, Q , of X . This means that $u \in W^{g-1}_{a-\Phi(\hat{R})} \ominus (-W^1) = -W^{g-2}_{-(a-\Phi(\hat{R})+k)}$. The proof of the lemma is now completed by observing that

$$-W^{g-2}_{-(a-\Phi(R)+k)} \subset W^{g-1}_b \quad .$$

The final lemmas of this section are restatements of well-known results in a convenient form.

Lemma 6: *Suppose* $W^1_a \not\subset -W^{g-1}_b$. *Then the intersection* $W^1_a \cap -W^{g-1}_b$ *consists of a finite series of points*, $u_1,\ldots,u_r$, *and we can assign positive multiplicities*, $m_1,\ldots,m_r$, *to these points in a unique manner such that*

$$\sum m_i = g \ ,$$

and

$$\sum m_i u_i = -(a+b) + ga \quad .$$

Proof: By the solvability of the Jacobi inversion problem, we can find a positive divisor, D , of degree g , such that $\Phi(D) = -(a+b)$. Let Q be any point of D . Then $D = D'+Q$,

where D' is of degree $(g-1)$ and

$$\phi(Q) + a = -(\phi(D') + b) \quad .$$

Hence $\phi(Q) + a$ is a point of the intersection. If $D = Q_1 + \dots + Q_g$, then $\phi(Q_i) + a$ is a point of the intersection for every i, and multiplicities may be assigned in accordance with the multiplicity of Q_i in D. Clearly

$$\sum (\phi(Q_i) + a) = -(a+b) + ga \quad .$$

To show uniqueness, assume that u is a point of the intersection. Then there are divisors Q and $\hat{Q}$ of degrees 1 and $(g-1)$, respectively such that

$$\phi(Q) + a = -(\phi(\hat{Q}) + b) \quad ,$$

or $\phi(Q+\hat{Q}) = -(a+b)$. If $Q+\hat{Q} = D$, u is one of the points already determined. If not, then $Q+\hat{Q} \sim D$, and $\dim(-D) \geq 2$. Hence we can find a positive divisor, $\hat{R}$, of degree g, containing an arbitrary point $R \varepsilon X$, such that $\phi(\hat{R}) = -(a+b)$. By the preceding argument, $\phi(R) + a$ is a point of the intersection. Since R is arbitrary, it follows that $W_a^1 \subset W_b^{g-1}$ contrary to hypothesis. This completes the proof.

Suppose now that, in the intersection $W_a^1 \cap -W_b^{g-1}$, we keep a fixed and let b vary over some set, $T \subset J(X)$. Suppose further that for all $b \varepsilon T$, $W_a^1 \not\subset -W_b^{g-1}$, and let $u_1(b),\dots,u_g(b)$ be the points of the intersection $W_a^1 \cap -W_b^{g-1}$. By virtue of Lemma 6 we have

$$\sum u_i(b) = -(a+b) + ga ,$$

or

$$b = (g-1)a - \sum u_i(b) .$$

Suppose finally that, as b varies over T, the points, $u_{r+1}(b),\ldots,u_g(b)$ remain fixed. Then $\sum u_i(b)$ varies over W_d^r, $d = u_{r+1}(b) + \ldots + u_g(b)$, and

$$T_{-(g-1)a} \subset -W_d^r .$$

3. TORELLI'S THEOREM

We have shown that the Jacobian variety, $J(X)$, is completely determined by a period matrix, $(\pi iE; A)$ of X. The converse is not true, since it is perfectly possible that the column vectors of another Riemann matrix, $(\pi ie; A')$, may generate the same translation group as those of $(\pi iE; A)$.

A theorem of Riemann's, which we shall prove in the appendix, shows that the period matrix, $(\pi iE; A)$, determines not only the Jacobian variety, $J(X)$, but also the class of translates of W^{g-1}. More precisely, Riemann's theorem identifies W^{g-1} as the set of zeros of a certain (multiplicative) holomorphic function on $J(X)$. By Lemma 4 it follows that all the sets $\pm W_a^r$ are given as common zeros of holomorphic functions.

We therefore define a *canonically polarized Jacobian variety* to be a pair $\left(J(X), \left\{W_a^{g-1}\right\}\right)$, and may now state Torelli's theorem in the following form:

Theorem: *A canonically polarized Jacobian variety*, $\left(J(X), \left\{W_a^{g-1}\right\}\right)$, *determines the canonically imbedded curve*, W^1, *up to a translation and reflection*.

Proof: Let X and Y be two closed Riemann surfaces of genus g, and let W^r (resp V^r) be the image under the canonical mapping Φ (resp ψ) of positive divisors of degree r on X (resp Y) into $J(X)$ (resp $J(Y)$).

Assume $\left(J(X), \left\{W_a^{g-1}\right\}\right) = \left(J(Y), \left\{V_b^{g-1}\right\}\right)$, i.e., assume that X and Y have the same canonically polarized Jacobian variety. We shall show that W^1 is a translate of $\pm V^1$.

We proceed by a reductio ad absurdum. Suppose $\pm W^1 \neq V_a^1$ for all $a \varepsilon J(X)$. Let r be the smallest integer ≥ 1 for which some inclusion of the form

$$W^1 \subset V_a^{r+1}, \quad -W^1 \subset V_a^{r+1}, \quad V^1 \subset W_a^{r+1}, \quad \text{or} \quad -V^1 \subset W_a^{r+1}$$

holds. Clearly $r < g-2$.

Assume, changing the notation if necessary, that $V^1 \subset W_a^{r+1}$, and let $b = a+x-y$, where $x \varepsilon W^1$, and $y \varepsilon W^{g-1-r}$. Set $c = x+a = y+b$. Then, unless $W_a^{r+1} \subset W_b^{g-1}$, we have, by Lemma 5 $W_a^{r+1} \cap W_b^{g-1} = W_c^r \cup S$, where $S = W_a^{r+1} \cap -W^{g-2}_{-(a+k-y)}$. Under these conditions, W_c^r depends only on the choice of x, and S depends only on the choice of y.

We shall first show that, in general, $V^1 \not\subset W_b^{g-1}$. Let x be fixed, and let y vary over W^{g-1-r}. Then b varies over $-W^{g-1-r}_{-(a+x)}$. If $V^1 \subset W_b^{g-1}$ for all $b \in -W^{g-1-r}_{-(a+x)}$, then $V^1 \subset W^{g-1} \ominus W^{g-1-r}_{-(a+x)}$, whence $V^1 \subset W^r_{a+x}$, contrary to hypothesis.

Since $W_b^{g-1} = V_{b+d}^{g-1}$ for some d, we have $V^1 \subset W_b^{g-1} \iff b \in -V_d^{g-2}$, and the set of $b \in -W^{g-1-r}_{-(a+x)}$ for which $V^1 \subset W_b^{g-1}$ is given by

$$-V_d^{g-2} \cap -W^{g-1-r}_{-(a+x)} .$$

Since there are points in $-W^{g-1-r}_{-(a+x)}$ which do not lie in this intersection, it follows that the intersection is a lower-dimensional subset of $-W^{g-1-r}_{-(a+x)}$.

For y in a dense subset of W^{g-1-r}, we have accordingly

$$V^1 \not\subset W_b^{g-1} ,$$

and therefore also

$$W_a^{r+1} \not\subset W_b^{g-1} .$$

It follows from Lemma 5 that for such a choice of y, $V^1 \cap W_b^{g-1}$ contains a finite number of points, and we must have, since $V^1 \subset W_a^{r+1}$,

$$V^1 \cap W_b^{g-1} = \left(V^1 \cap W_c^r\right) \cup (V^1 \cap S) .$$

If we keep x fixed, and let y vary, only the points in $V^1 \cup S$ can change, since W^r_c remains fixed. We show first that there is at most one point in the intersection $V^1 \cap W^r_c$. If not, there are at least two fixed points in the intersection $V^1 \cap W^{g-1}_b$ as y varies over a dense subset of W^{g-1-r} . Hence, by the remarks following Lemma 6, this dense subset is included in a translate of $-V^{g-2}$, and W^{g-1-r} is a subset of some translate of $-V^{g-2}$. But if $W^{g-1-r} \subset -V^{g-2}$, then

$$V^1 = V^{g-1} \ominus V^{g-2} \subset W^{g-1} \ominus -W^{g-1-r} = -W^r_{-k} ,$$

ignoring subscripts. This contradicts the assumption on r .

Now, if we keep y fixed and vary x , at least one point of the intersection must change, and this point cannot lie in $V^1 \cap S$, and it follows that exactly one point is associated with $V^1 \cap W^r_c$.

We must now examine the possibility that this point might occur in the intersection $V^1 \cap W^{g-1}_b$ with multiplicity > 1 . It is clear from the preceding argument that this cannot happen for all y , and hence we may select a y for which the point occurs with multiplicity one. Varying x over a sufficiently small neighborhood, it is then easily seen from Lemma 6 that this yields an inclusion of that neighborhood in W^1 into some neighborhood in a translate of V^1 . But then it is easy to see that we must have an inclusion of V^1 in some translate of W^1 , and this would complete the proof.

We shall, however, give a different argument which avoids any appeal to continuity, and which therefore is valid in the abstract case as well. From the earlier argument, it is clear that those choices of y for which the point occurs with multiplicity > 1 is given by an intersection of W^{g-1-r} with some translate of $-V^{g-2}$. By the irreducibility of W^{g-1-r}, this intersection must be a lower-dimensional subset of W^{g-1-r}. Hence, given any finite sequence, $x_1, \ldots, x_n$, we can find a y such that the point only appears with multiplicity 1, for all i. Since the points of $V^1 \cap S$ are fixed, their multiplicities may assume only a finite number of distinct values, and taking n large enough we shall find points x_i, x_j such that

$$V^1 \cap W^{g-1}_{a+x_i-y} = \{v_1, v_2, \ldots v_s\}$$

$$V^1 \cap W^{g-1}_{a+x_j-y} = \{\hat{v}_1, v_2, \ldots, v_s\}$$

where $v_2, \ldots, v_s$ occur with the same multiplicities in both intersections, and $v_1, \hat{v}_1$ occur with multiplicity 1.

Then $x_i - x_j = v_1 - \hat{v}_1$, and it follows that V^1 has two distinct points in common with some translate of W^1, say W^1_e. But then

$$W^{g-2}_{-e} = W^{g-1} \ominus W^1_e \subset V^{g-1}_{d-x_i} \cap V^{g-1}_{d-x_j} .$$

By Lemma 5, the right-hand intersection decomposes into a union

of a translate of V^{g-2} and a translate of $-V^{g-2}$, and hence W^{g-2} is included in a translate of V^{g-2} or of $-V^{g-2}$. Since $W^{g-1} \ominus W^{g-2} = W^1$, this yields an inclusion of V^1 in a translate of W^1 or of $-W^1$, contrary to assumption. This completes our proof.

APPENDIX[1]

Since, in the period matrix $(\pi i E; A)$, A has a negative definite real part, it is easy to show the uniform convergence on compact subsets of the series

$$\theta(u; A) = \sum_{m} \exp({}^t m(Am+2u)) \quad ,$$

where ${}^t u = (u^1, \ldots, u^g) \in C^g$, and ${}^t m = (m^1, \ldots, m^g)$ is a g-vector with integral components. The sum is taken over all such m. The function $\theta(u; A)$ will be referred to as a first-order theta-function formed with A. Any translate of this function is also called a theta-function of the first-order.

The theta-functions have the following, easily verifiable, periodicity behavior:

$$\theta(u+\pi i n; A) = \theta(u; A) \quad ,$$

and

$$\theta(u+An; A) = \theta(u; A) \exp(-{}^t n(2u+An)) \quad ,$$

where n is any g-vector with integral components. Hence they are multiplicative functions on $J(X)$.

Riemann showed that if $(\pi i E, A)$ is a period matrix with

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[1] General references: [2] and [7].

respect to a canonical homology basis on a closed Riemann surface, then the following amazing characterization of the zeros of $\theta(u; A)$ may be given

Riemann's Theorem: *Let* X *be a closed Riemann surface of genus* g . *Let* γ *be a canonical homology basis, and let* ω *be a basis for the Abelian differentials of the first kind. Let* $(\gamma, \omega) = (\pi i E; A)$. *Then, given a point,* P , *on* X , *there are constants,* c^i , *depending on* P , *such that* $\theta(u; A) = 0$ *iff*

$$u^i = -\sum_{j=1}^{g-1} \int_P^{Q_j} \omega^i + c^i$$

for some $(g-1)$-*tuple of points,* Q_i , *on* X .

A detailed proof of this theorem will be found in [7]. Here we shall merely indicate the idea of the proof.

We assume X represented by a fundamental polygon in the unit disc, and set

$$u^i(Q) = \int_P^Q \omega^i \quad ,$$

where Q varies over the fundamental polygon. Then $\theta(u(Q); A)$ is a well-defined function on the fundamental polygon, and, evaluating the integral around the polygon of its logarithmic derivative, we find that unless θ vanishes identically on X , it has precisely g zeros (counting multiplicities) in the polygon. The same is true for any translate, $\theta(u+b; A)$. Let

$Q_1(b),\ldots,Q_g(b)$ be the zeros of $\theta(u+b; A)$. Evaluating the integrals around the polygon of the products of $u^i(Q)$ with the logarithmic derivative of

$$\frac{\theta(u+b; A)}{\theta(u+b'; A)} ,$$

we find that $\sum u(Q_i(b)) - \sum u(Q_i(b')) = -b+b'$. Hence $\sum u(Q_i(b)) + b$ is a constant, c , depending only on P .

Now consider the function

$$\theta\left(u(Q) + c - \sum u(Q_i(b)); A\right) .$$

This function vanishes for $Q = Q_i$, and then its argument is

$$c - \sum_{j \neq i} u(Q_j(b)) .$$

Take now an arbitrary $(g-1)$-tuple $Q_2,\ldots,Q_g$, and let

$$b = c - \sum_{i=1}^{g} u(Q_i) ,$$

where Q_1 is some additional point on X . Since the Jacobi inversion problem is, in general, uniquely solvable, it follows that the above representation of b is, in general, unique. But the above equation holds for the zeros of $\theta(u(Q) + b; A)$, and hence the Q_i must be its zeros. By the preceding argument we have

$$\theta\Big(c - \sum_{2}^{g} u(Q_i)\Big) = 0 \quad .$$

The exceptional cases, where the representation is not unique, are easily shown to be limiting cases of unexceptional ones, and may be treated by a continuity argument. In fact, when the representation is not unique, it may be seen that $\theta(u(Q) + b; A)$ vanishes identically on X .

By virtue of Riemann's theorem it follows that the image in C^g under the map

$$u^i = - \sum_{j=1}^{g-1} \int_P^{Q_j} \omega^i \quad ,$$

of $(g-1)$-tuples of points on X , is determined up to a translation by the period matrix $(\pi iE; A)$, since it is the set of zeros of $\theta(u; A)$.

Thus the sets W_a^{g-1} are determined as the zero manifolds of first-order theta-functions.

We shall now draw an interesting conclusion from the irreducibility of W^{g-1} . Assume that, in the period matrix, $(\pi iE,A)$ A decomposes into a direct sum

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where A_1 is an $n \times n$ matrix. It is readily seen from the definition that in this case

$$\theta(u; A) = \theta(v; A_1)\theta(w; A_2) \quad ,$$

where v is an n-vector, w is a $(g-n)$-vector, and ${}^t u = (v^1, \ldots, v^n, w^1, \ldots, w^{g-n})$. Now, since W_a^{g-1} is the zero manifold of $\theta(u; A)$ for some a, it follows that $\theta(v; A_1)$ vanishes identically on a full neighborhood of W_a^{g-1}, but not everywhere, since the zeros of $\theta(v; A_1)$ and $\theta(w; A_2)$ do not coincide. This contradicts the irreducibility of W_a^{g-1}, and we conclude

Theorem: If $(\pi i E; A)$ is a period matrix with respect to a canonical homology basis on a closed Riemann surface, then A is not symplectically equivalent to a direct sum.

This generalizes and contains a previous result due to Wirtinger [3].

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